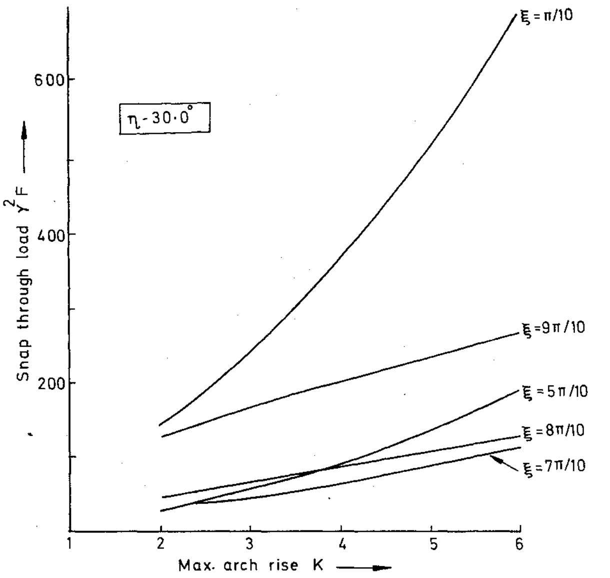

Fig. 2 $\gamma^2 F$ **vs** k **for various load positions (**$\eta = 30°$**).**

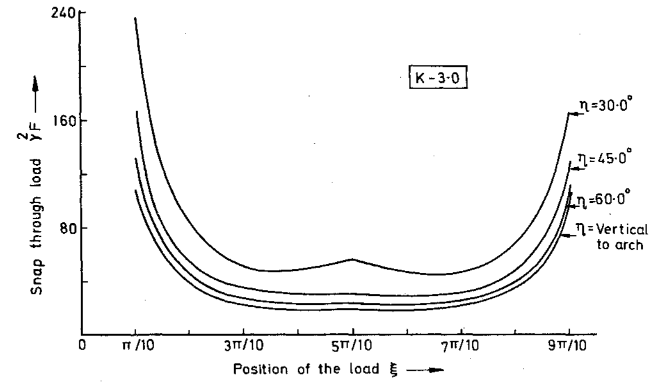

Fig. 3 $\gamma^2 F$ **vs position of the load for various angle of inclination of the load** η**.**

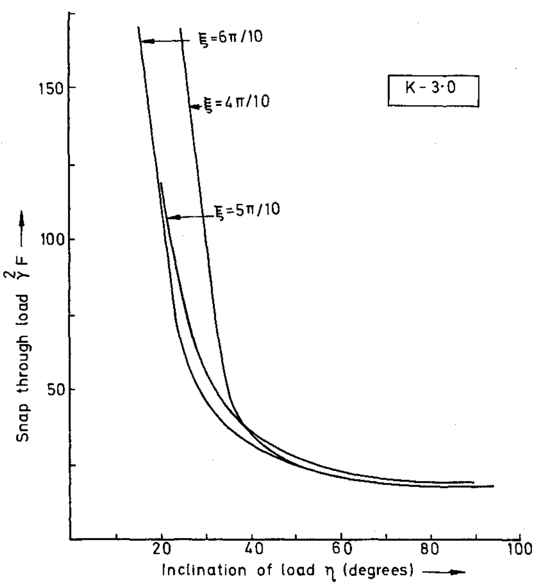

Fig. 4 $\gamma^2 F$ **vs** η **for various position of the load.**

center of the arch whereas this is not the case when the load acts at an inclination. The variation of the snap-through load with η for various positions of the load corresponding to the case when $k = 3$ is shown in Fig. 4. The snap-through load remains nearly a constant after a certain value of the inclination η.

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A Modified Gradient Technique for Solving Boundary and Initial Value Problems

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Nomenclature

a_k = polynomial constant coefficients
E_i = error of the ith equation
M = polynomial degree
N = number of strips for boundary-value problems
r = number of boundary conditions
x_j = generalized state variables

Introduction

WHEN applying a numerical scheme to a system of nonlinear differential equations, they usually reduce to a system of nonlinear algebraic equations. There are a number of methods for solving nonlinear equations, such as, the shoot and hunt scheme, the perturbation method,[1,2] the method of quasilinearization,[3,4] and the method of accelerated successive replacements.[5]

The shoot and hunt method has the drawback of its sensitivity in some cases to the assumed initial values. The perturbation method depends mainly on the existence of small nonlinearity. For highly nonlinear equations, the method is questionable. The method of quasilinearization can not be applied successfully for systems producing ill conditioned matrices. On the other hand, the method of accelerated successive replacements alleviates the difficulties encountered in the other three methods. However, the number of iterations required to achieve a certain accuracy is considerably larger.[5]

The present analysis is intended to provide a simple and fast, yet accurate, method for solving boundary and initial value problems. Two applications are shown, the Blasius boundary-layer problem, which reduces to the solution of a highly nonlinear two point boundary value differential equation, and the problem of hypersonic flow over cones based on the hypersonic small disturbance theory.

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Formulation

Consider a nonlinear differential equation governing the variation of the function $F(\eta)$, which may be written in a functional form as

$$\psi(F, F', F'', F''', \ldots) = 0 \tag{1}$$

with boundary conditions,

$$g_L(F, F', \ldots) = 0, \qquad L = 1, 2, \ldots, r \tag{2}$$

where primes denote differentiation with respect to η.

To start the numerical solution, represent the function $F(\eta)$ by M degree polynomial

$$F(\eta) = \sum_{k=0}^{M} a_k \eta^k \tag{3}$$

Divide the η domain into N equal strips, each of length $\delta\eta$, i.e.

$$\eta_e = N(\delta\eta) \tag{4}$$

The values of M and N are inter-related as will be seen. Substituting Eq. (3) into Eq. (1) and apply at nodal points $\eta = n(\delta\eta)$; $n = 1, 2, \ldots, N-1$, thus yielding a system of $(N-1)$ equations

$$\psi_n(a_0, a_1, \ldots, a_M, \delta\eta) = 0; \qquad n = 1, 2, \ldots, N-1 \tag{5}$$

For many physical problems, such as the boundary-layer problem, the outer limit of η, i.e. η_e is not known a priori and is to be determined from the solution. Considering this general case, the number of unknowns $(a_0, a_1, \ldots, a_M, \delta\eta)$ is $(M+2)$ and the number of equations together with boundary conditions are $(N-1+r)$. Therefore, the problem is mathematically determinate when

$$N = M + 3 - r \tag{6}$$

For strong varying functions, M must be large and consequently N is large.

Method of Solution

The system of nonlinear equations required to be solved is composed of Eqs. (2) and (5). Symbolically this system may be written as

$$\phi_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_j, \ldots, \bar{x}_{M+2}) = 0$$
$$i = 1, 2, \ldots, (N-1+r) \quad \text{and} \quad j = 1, 2, \ldots, (M+2) \tag{7}$$

which possess continuous partial derivatives with respect to the state variables $\bar{x}_j$'s. The state variable $\bar{x}_j$'s here represent the unknowns $(a_0, a_1, \ldots, a_M, \delta\eta)$ to be evaluated. Let's defined the barred state variables $\bar{x}_j$'s as those variables (unknowns) which satisfy exactly the available system of equations. For any other set x_j's, system (7) becomes

$$\phi_i(x_1, x_2, \ldots, x_j, \ldots, x_{M+2}) = E_i$$
$$i = 1, 2, \ldots, (N-1+r) \quad \text{and} \quad j = 1, 2, \ldots, M+2 \tag{8}$$

where E_i's are the errors introduced in the system of equations. The root mean square error (rmse) is thus

$$\text{rmse} = \left(\sum_{i=1}^{N-1+r} E_i^2 \right)^{1/2} \tag{9}$$

as any value of the state variables x_j's changes, or as the whole set x_j's change, rmse changes accordingly. If the state variables can be represented n an $(M+2)$ space, therefore, the displacement in this Euclidian space is

$$ds^2 = \sum_{j=1}^{M+2} dx_j^2 \tag{10}$$

The variation of rmse along the displacement ds becomes;

$$\frac{d}{ds}\left(\sum_{i=1}^{N-1+r} E_i^2 \right)^{1/2} = \sum_{j=1}^{M+2} \frac{d}{dx_j}\left[\sum_{i=1}^{N-1+r} \phi_i^2(x_1, \ldots, x_j, \ldots, x_{M+2}) \right]^{1/2} \frac{dx_j}{ds} \tag{11}$$

From Eqs. (8, 9, and 11) we get

$$\frac{d}{ds}\left(\sum_{i=1}^{N-1+r} \phi_i^2 \right)^{1/2} = \frac{1}{rmse} \sum_{j=1}^{M+2} \left(\sum_{i=1}^{N-1+r} \phi_i \frac{\partial \phi_i}{\partial x_j} \right) \frac{dx_j}{ds} \tag{12}$$

The values of x_j's will be closest to $\bar{x}_j$'s when rmse becomes minimum. Seeking a minimum values of rmse, we search for directions dx_j/ds which makes the left hand side of Eq. (12) negative. The most negative direction (steepest descent) may be found by applying the variational concept in the following manner:

Extremise the right hand side of Eq. (12) subject to the geometrical constraint (10). The augmented function is thus

$$Z \equiv \frac{1}{\text{rmse}} \sum_{j=1}^{M+2} \left(\sum_{i=1}^{N-1+r} \phi_i \frac{\partial \phi_i}{\partial x_j} \right) \frac{dx_j}{ds} + \lambda_0 \left[1 - \sum_{j=1}^{M+2} \left(\frac{dx_j}{ds} \right)^2 \right] \tag{13}$$

where λ_0 is a constant Lagrange multiplier.

From the necessary condition for an extremum of the augmented function Z with respect to dx_j/ds, we get

$$\frac{dx_j}{ds} = \frac{1}{2\lambda_0(\text{rmse})} \sum_{i=1}^{N-1+r} \phi_i \frac{\partial \phi_i}{\partial x_j} \tag{14}$$

Equation (14) gives the direction cosines of the optimum displacement required to achieve fastest convergence to minimum rmse. The Lagrange multiplier λ_0 is determined by comparing Eqs. (10) and (14). Such comparison will yield one value for λ_0 with two signs. Since we are seeking a minimum value of rmse, the positive sign will be dropped. Equation (14) is thus written in difference form, which may be represented conveniently as

$$x_j^{p+1} = x_j^p - \sum_{i=1}^{N-1+r} \phi_i \frac{\partial \phi_i}{\partial x_j} \Bigg/ \left[\sum_{j=1}^{M+2} \left(\sum_{i=1}^{N-1+r} \phi_i \frac{\partial \phi_i}{\partial x_j} \right)^2 \right]^{1/2} \tag{15}$$

where superscripts denote order of iterative process.

Application to Boundary Value Problem

The present method is applied to the simple Blasius problem, which is a boundary-value problem. The governing equation is

$$f''' + ff'' = 0 \tag{16}$$

where primes denote differentiations with respect to the independent variable η.

In the domain $o \leqq \eta \leqq \eta_e$, represent f' by a fourth degree polynomial, i.e.

$$f'(\eta) = \sum_{k=0}^{M} a_k \eta^k; \qquad M = 4 \tag{17}$$

where a_k's are the polynomial coefficients.

After dividing the distance η_e into N strips, at any nodal point n, Eq. (16) with the help of Eq. (17) becomes

$$\phi_i = \sum_{j=2}^{4} k(k-1)a_k(n\delta\eta)^{k-2} + \left[\sum_{k=0}^{4} \frac{a_k}{k+1}(n\delta\eta)^{k+1} \right] \times \left[\sum_{k=1}^{4} ka_k(n\delta\eta)^{k-1} \right] = 0 \qquad i = n = 1, 2, \ldots, N-1 \tag{18}$$

The boundary conditions are[6] ($f'(0) = 0$, $f'(\eta_e) = 2.0$, $f''(\eta_e) = 0.0002$)

$$\phi_4 \equiv a_0 = 0 \tag{19a}$$

$$\phi_5 \equiv \sum_{k=1}^{4} a_k(N\delta\eta)^k - 2.0 = 0 \tag{19b}$$

$$\phi_6 \equiv \sum_{k=1}^{4} ka_k(N\delta\eta)^{k-1} - 0.0002 = 0 \tag{19c}$$

To get the required number of strips N, we apply Eq. (6) with $M = 4$ and $r = 3$. The result is that $N = 4$. In the boundary-layer analysis, η_e is not known a priori, therefore, to determine it, $\delta\eta$ will be considered as an additional unknown. The number of unknowns is thus six, $(a_0, a_1, \ldots, a_4, \delta\eta)$. The a_k's and $\delta\eta$ are the state variables denoted by x_j's in the previous section. To start the numerical solution an initial guess for the state variables is required. The linear profile ($a_1 = 1$, $a_2 = a_3 = a_4 = 0$) is taken together with $\delta\eta = 1$ to be the starting values.

The problem was coded in FORTRAN IV and run on the ICL 1905 E Computer of Cairo University. The values of f, f' and f'' were obtained at the three nodal points. Comparison between values calculated by the present method and those of Ref. 6 is shown in Table 1.

Table 1 Comparison of values of f, f' and f''

η	f		f'		f''	
	Present	Ref. 6	Present	Ref. 6	Present	Ref. 6
1.73	0.47591	0.47597	0.54778	0.54776	0.28897	0.28893
3.46	1.77701	1.77706	0.92169	0.91267	0.10679	0.10678
5.19	3.48062	3.48064	0.99367	0.99565	0.01159	0.01157

The running mill time was about 1.2 sec, compared to 4.3 sec (to get the same accuracy) by using finite difference approach.

Application to Initial Value Problem

Initial value problems can also be dealt with by the present method. Let's consider the equation obtained by the hypersonic small disturbance theory for the perturbed potential flow over a cone in the region between the shock wave and the body surface. The equation reads[7]

$$4f^2f'' - 2ff'^2 = \gamma\omega \frac{f'^{(\gamma+1)}}{\theta^{(\gamma-1)}}(f'' - f'/\theta) \tag{20}$$

where f is the nondimensional stream function, θ is the conical parameter (independent variable), γ is the ratio of specific heats, and ω is a known function of γ and K_s (where K_s is the shock wave hypersonic similarity parameter).[7]

In Eq. (20) primes denote differentiations with respect to θ. Equation (20) is to be solved subject to the initial conditions

$$f(1) = \tfrac{1}{2} \quad \text{and} \quad f'(1) = (\gamma+1)K_s^2/[2+(\gamma-1)K_s^2] \tag{21}$$

The solution starts from the shock wave ($\theta = 1$) and proceeds till f goes to zero (the cone surface). Making use of the linear interpolations

$$f''(\theta) = [f'(\theta) - f'(\theta-\Delta\theta)]/\Delta\theta \tag{22a}$$

$$f'(\theta) = [f(\theta) - f(\theta-\Delta\theta)]/\Delta\theta \tag{22b}$$

where $\Delta\theta$ is a small internal, whose magnitude depends upon the accuracy required. Denoting

$$x_1 = f(\theta); \quad \bar{x}_1 = f(\theta - \Delta\theta)$$

$$x_2 = f'(\theta); \quad \bar{x}_2 = f'(\theta - \Delta\theta)$$

Equations (20) and (22b) thus become

$$\phi_1 = 4x_1^2(x_2 - \bar{x}_2) - 2\Delta\theta x_1 x_2^2 - \frac{\gamma\omega}{\theta^{(\gamma-1)}} x_2^{(\gamma+1)} \times \left(x_2 - \bar{x}_2 - -\frac{\Delta\theta}{\theta} x_2\right) \tag{23}$$

$$\phi_2 = x_1 - \bar{x}_1 - \Delta\theta x_2$$

subject to the initial conditions given by Eq. (21), where $x_1{}^0 \equiv f(1) = \frac{1}{2}$ and $x_2{}^0 \equiv f'(1)$.

The preceding system of equations has been solved by the present method. Taking $\Delta\theta = 0.02$, the running mill time on the ICL 1905 E Computer of Cairo University (for $K_s = 1.58$) is 0.1 sec, compared to 0.25 sec by using the Runge-Kutta scheme. The obtained value of the pressure coefficient (which is a function of f') is the same as in Ref. 7.

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A Correlation of Freejet Data

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Introduction

THE subject for discussion is the two-dimensional, underexpanded, freejet flowing into a static medium. Interest in freejet studies has been prompted by modeling studies of interacting jet controls.[1] These studies have determined that the interacting jet shock height scales the two-dimensional jet interaction flowfield, and further, a direct correspondence exists between freejet shock heights and those interaction jets.[1] An earlier correlation study of freejet shock heights[2] was based upon axisymmetric nozzle data, both sonic and supersonic, and upon two-dimensional sonic nozzle data. Since then, data which include the supersonic slot nozzle case have been obtained. These results modify the earlier reported correlation. Several graphical solutions by the method of characteristics are also reported.

Experimental Conditions

The experimental apparatus included a slot nozzle mounted transversely between two glass-ported side plates. This assembly was placed in the test section of the wind tunnel which served as a convenient, low-pressure, test cell. The wind-tunnel vacuum pumping plant was used to maintain a constant reservoir pressure, P_b, (surrounding the jet plume). Test conditions and photographs were recorded for jet strengths, P_{0j}/P_∞, ranging from 29.4 to 915.6. Two supersonic jet nozzles were tested with Mach numbers of 2.89 and 2.99 and throat widths of 0.0335 in. and 0.0204 in. respectively. Three sonic jet nozzles of various slot widths were also tested as a check on previously published results.[2]

Experimental Results

An entirely unexpected experimental result was the observed absence of the centerline normal shock wave for the supersonic jet case. Figures 1a, 1b, and 1c show typical photographs of the supersonic jet shock structure. Additional photographic data are found in Ref. 3. Figure 1d shows a typical graphical solution by the method of characteristics. The observed differences between sonic and supersonic freejet plume shapes are shown sketched in Fig. 2a. Embedded within the jet plume are inter-

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